Contributions to Discrete Mathematics

Volume 14, Number 1, Pages 46–54
ISSN 1715-0868

EULERIAN POLYNOMIALS AND POLYNOMIAL CONGRUENCES

KAZUKI IIJIMA, KYOUHEI SASAKI, YUUKI TAKAHASHI,
AND MASAHIKO YOSHINAGA

Abstract. We prove that the Eulerian polynomial satisfies certain polynomial congruences. Furthermore, these congruences characterize the Eulerian polynomial.

1. Introduction

The Eulerian polynomial $A_\ell(x)$ ($\ell \geq 1$) was introduced by Euler in the study of sums of powers [5]. In this paper, we define the Eulerian polynomial $A_\ell(x)$ as the numerator of the rational function

$$F_\ell(x) = \sum_{k=1}^{\infty} k^\ell x^k = \left(x\frac{d}{dx}\right)^\ell \frac{1}{1-x} = \frac{A_\ell(x)}{(1-x)^{\ell+1}}. \tag{1.1}$$

The first few examples are $A_1(x) = x, A_2(x) = x + x^2, A_3(x) = x + 4x^2 + x^3, A_4(x) = x + 11x^2 + 11x^3 + x^4$, etc. The Eulerian polynomial $A_\ell(x)$ is a monic of degree ℓ with positive integer coefficients. Write $A_\ell(x) = \sum_{k=1}^{\ell} A(\ell, k)x^k$. The coefficient $A(\ell, k)$ is called an Eulerian number.

In the 1950s, Riordan [10] discovered a combinatorial interpretation of Eulerian numbers in terms of descents and ascents of permutations, and Carlitz [3] defined q-Eulerian numbers. Since then, Eulerian numbers are actively studied in enumerative combinatorics. (See [4, 11, 8].)

Another combinatorial application of the Eulerian polynomial was found in the theory of hyperplane arrangements [15, 14]. The characteristic polynomial of the so-called Linial arrangement [9] can be expressed in terms of the root system generalization of Eulerian polynomials introduced by Lam and Postnikov [6]. The comparison of expressions in [9] and [15] yields the following.

Received by the editors June 15, 2017, and in revised form January 13, 2019.

2000 *Mathematics Subject Classification.* 05A15, 05E99.

Key words and phrases. Eulerian polynomials, Hyperplane arrangements.

The authors thank Mr. Rei Yoshida for several discussions about the contents of this paper. The authors also thank Prof. Richard Stanley for noticing [12]. M. Yoshinaga was partially supported by JSPS KAKENHI Grant Number JP25400060, JP15KK0144, and JP16K13741.

Theorem 1.1 ([15, Proposition 5.5])**.** *For $\ell, m \geq 2$, the Eulerian polynomial $A_\ell(x)$ satisfies the following*

$$A_\ell(x^m) \equiv \left(\frac{1+x+x^2+\cdots+x^{m-1}}{m}\right)^{\ell+1} A_\ell(x) \mod (x-1)^{\ell+1}. \tag{1.2}$$

The purpose of this paper is two-fold. First, we give a direct and simpler proof of Theorem 1.1. Second, we prove the converse of the above theorem. Namely, the congruence (1.2) characterizes the Eulerian polynomial as follows.

Theorem 1.2. *Let $f(x)$ be a monic of degree ℓ. Then, $f(x) = A_\ell(x)$ if and only if the congruence (1.2) holds for some $m \geq 2$. (See Theorem 5.1.)*

The remainder of this paper is organized as follows. After recalling classical results on Eulerian polynomials in §2, we briefly describe in §3 the proof of the congruence (1.2) in [15] that is based on the expression of characteristic polynomials of Linial hyperplane arrangements. In §4, we give a direct proof of the congruence. In §5, we give the proof of Theorem 1.2.

Remark. The right-hand side of (1.2) is discussed also in [12, Proposition 2.5].

2. Brief review of Eulerian polynomials

In this section, we recall classical results on the Eulerian polynomial and the Eulerian numbers $A(\ell, k)$. By definition (1.1), the Eulerian polynomial $A_\ell(x)$ satisfies the relation

$$\frac{A_\ell(x)}{(1-x)^{\ell+1}} = x\frac{d}{dx}\frac{A_{\ell-1}(x)}{(1-x)^\ell},$$

which yields the following recursive relation.

$$A(\ell,k) = k\cdot A(\ell-1,k) + (\ell-k+1)\cdot A(\ell-1,k-1). \tag{2.1}$$

Consider the coordinate change $w = \frac{1}{x}$. Then, the Euler operator is transformed as $x\frac{d}{dx} = -w\frac{d}{dw}$. The direct computation using the relation $\frac{1}{1-\frac{1}{w}} = 1 - \frac{1}{1-w}$ yields $x^{\ell+1}A_\ell(\frac{1}{x}) = A_\ell(x)$. Equivalently, $A(\ell,k) = A(\ell, \ell+1-k)$.

Definition (1.1) is also equivalent to

$$A_\ell(x) = (1-x)^{\ell+1}\cdot\sum_{k=0}^{\infty} k^\ell x^k \tag{2.2}$$

and

$$\sum_{k=0}^{\infty} k^\ell x^k = A_\ell(x)\cdot\sum_{k=0}^{\infty}(-x)^k\binom{-\ell-1}{k}. \tag{2.3}$$

Then, (2.2) yields

$$A(\ell,k)=\sum_{j=0}^{k}(-1)^j\binom{\ell+1}{j}(k-j)^\ell, \tag{2.4}$$

and (2.3) yields

$$k^\ell=\sum_{j=1}^{\ell}A(\ell,j)\binom{k+\ell-j}{\ell}. \tag{2.5}$$

Note that both sides of (2.5) are polynomials of degree ℓ in k, and it holds for any $k>0$. Hence the equality holds at the level of polynomials in t. Thus, we have

$$t^\ell=\sum_{j=1}^{\ell}A(\ell,j)\binom{t+\ell-j}{\ell}, \tag{2.6}$$

which is called the Worpitzky identity [13]. Using the shift operator $S : t\longmapsto t-1$ (see §3), the Worpitzky identity can be written as

$$t^\ell=A_\ell(S)\binom{t+\ell}{\ell}. \tag{2.7}$$

Next, we consider exponential generating series of $A_\ell(x)$, and describe relations with Bernoulli numbers. First, using (2.2), we have

$$\begin{aligned}\sum_{\ell=0}^{\infty}\frac{A_\ell(x)}{\ell!}t^\ell&=\sum_{\ell=0}^{\infty}\frac{t^\ell}{\ell!}(1-x)^{\ell+1}\sum_{n=0}^{\infty}n^\ell x^n\\&=(1-x)\sum_{n=0}^{\infty}x^n e^{nt(1-x)}\\&=\frac{1-x}{1-xe^{t(1-x)}}.\end{aligned} \tag{2.8}$$

Replacing x by -1 in (2.8), we have

$$\sum_{\ell=0}^{\infty}\frac{A_\ell(-1)}{\ell!}t^\ell=\frac{2}{1+e^{2t}}. \tag{2.9}$$

Recall that the Bernoulli polynomial $B_\ell(x)$ is defined by

$$\sum_{\ell=0}^{\infty}\frac{B_\ell(x)}{\ell!}t^\ell=\frac{te^{xt}}{e^t-1}, \tag{2.10}$$

($B_0(x)=1, B_1(x)=x-\frac{1}{2}, B_2(x)=x^2-x+\frac{1}{6}, B_3(x)=x^3-\frac{3}{2}x^2+\frac{1}{2}x, B_4(x)=x^4-2x^3+x^2-\frac{1}{30},\cdots$) and the constant term $B_\ell(0)$ is called the Bernoulli number. Replacing x by 0 and t by at with $a\in\mathbb{C}$ in (2.10), we have

$$\sum_{\ell=0}^{\infty}\frac{B_\ell(0)}{\ell!}(at)^\ell=\frac{at}{e^{at}-1}. \tag{2.11}$$

Using the identity $\frac{2t}{e^{2t}+1} = \frac{2t}{e^{2t}-1} - \frac{4t}{e^{4t}-1}$, the Bernoulli number $B_\ell(0)$ can be expressed as

$$B_\ell(0) = \frac{\ell}{2^\ell(1-2^\ell)} A_{\ell-1}(-1). \tag{2.12}$$

There is another relation between Eulerian polynomials and Bernoulli polynomials. Let $\ell > 0$. Using (2.5) and the famous formula $\sum_{x=0}^{N-1} x^\ell = \frac{B_{\ell+1}(N)-B_{\ell+1}(0)}{\ell+1}$, we have

$$B_{\ell+1}(N) - B_{\ell+1}(0) = (\ell+1) \cdot \sum_{k=1}^{\ell} A(\ell,k) \binom{\ell+N-k}{\ell+1}. \tag{2.13}$$

With the shift operator S, (2.13) can also be expressed as

$$B_{\ell+1}(t) - B_{\ell+1}(0) = (\ell+1) A_\ell(S) \binom{t+\ell}{\ell+1}. \tag{2.14}$$

(This formula appeared in [13, page 209] as "The second form of Bernoulli function.")

3. Background on hyperplane arrangements

In this section, we recall the proof of the congruence (1.2) presented in [15].

Let $\mathcal{A} = \{H_1, \dots, H_k\}$ be a finite set of affine hyperplanes in a vector space V. We denote the set of all intersections of $\mathcal{A}$ by $L(\mathcal{A}) = \{\cap S \mid S \subset \mathcal{A}\}$. The set $L(\mathcal{A})$ is partially ordered by reverse inclusion, which has a unique minimal element $\hat{0} = V$. The characteristic polynomial of $\mathcal{A}$ is defined by

$$\chi(A, q) = \sum_{X \in L(\mathcal{A})} \mu(X) q^{\dim X},$$

where μ is the Möbius function on $L(\mathcal{A})$, defined by

$$\mu(X) = \begin{cases} 1, & \text{if } X = \hat{0} \\ -\sum_{Y<X} \mu(Y), & \text{otherwise.} \end{cases}$$

Let $V = \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{\ell+1} \mid \sum x_i = 0\} \subset \mathbb{R}^{\ell+1}$. For integers $0 \le i < j \le \ell$ and $s \in \mathbb{Z}$, denote by $H_{ij,s}$ the affine hyperplane $\{(x_0, \dots, x_\ell) \in V \mid x_i - x_j = s\}$.

Let $m \ge 1$ be a positive integer. The arrangement

$$\mathcal{L}^m = \{H_{ij,s} \mid 0 \le i < j \le \ell, 1 \le s \le m\}$$

is called the (extended) Linial arrangement (of type A_ℓ). The Linial arrangement has several intriguing enumerative properties [9]. Postnikov and Stanley [9] (see also [1]) gave the following expression for the characteristic polynomial $\chi(\mathcal{L}^m, t)$.

$$\chi(\mathcal{L}^m, t) = \left(\frac{1 + S + S^2 + \cdots + S^m}{m+1} \right)^{\ell+1} t^\ell, \tag{3.1}$$

where S acts on a function $f(t)$ by $Sf(t) = f(t-1)$ (naturally $S^k f(t) = f(t-k)$) as the shift operator. Using the Worpitzky identity (2.7), (3.1) can be written as

$$\chi(\mathcal{L}^m, t) = \left(\frac{1+S+S^2+\cdots+S^m}{m+1}\right)^{\ell+1} A_\ell(S)\binom{t+\ell}{\ell}. \tag{3.2}$$

On the other hand, using the lattice points interpretation of the Worpitzky identity, the following formula was obtained in [15]

$$\chi(\mathcal{L}^m, t) = A_\ell(S^{m+1})\binom{t+\ell}{\ell}. \tag{3.3}$$

The formulas (3.2) and (3.3) imply that the operator

$$\left(\frac{1+S+S^2+\cdots+S^m}{m+1}\right)^{\ell+1} A_\ell(S) - A_\ell(S^{m+1}) \tag{3.4}$$

annihilates the polynomial $\binom{t+\ell}{\ell}$ of degree ℓ, which means that (3.4) is divisible by $(S-1)^{\ell+1}$ (see [15, Prop. 2.8]). Hence the congruence (1.2) follows.

4. Direct proof of the congruence

4.1. **Special case: $m = 2$.** We first handle the case $m = 2$. By considering $F_\ell(x) + F_\ell(-x)$, it is easily seen that the formal power series $F_\ell(x) = \sum_{k=1}^{\infty} k^\ell x^k$ satisfies

$$F_\ell(x) - 2^{\ell+1}F_\ell(x^2) = -F_\ell(-x). \tag{4.1}$$

Using the Eulerian polynomial, (4.1) can be written as

$$(1+x)^{\ell+1}\cdot A_\ell(x) - 2^{\ell+1}\cdot A_\ell(x^2) = -(1-x)^{\ell+1}\cdot A_\ell(-x), \tag{4.2}$$

which implies the congruence (1.2) for $m = 2$.

Remark. Substituting formally $x = 1$ into (4.1), we obtain the formula "$F_\ell(1) = \frac{A_\ell(-1)}{2^{\ell+1}(2^{\ell+1}-1)}$." Then, (2.12) implies "$F_\ell(1) = -\frac{B_{\ell+1}(0)}{\ell+1}$", which gives the correct value $\zeta(-\ell) = -\frac{B_{\ell+1}(0)}{\ell+1}$ of the Riemann zeta function for $\ell \geq 1$.

4.2. **General case.** Let $m \geq 2$. Denote by $\zeta_m = e^{2\pi\sqrt{-1}/m}$ the primitive mth root of unity. We will use the following fact

$$\sum_{j=1}^{m-1} \zeta_m^{jk} = \begin{cases} m-1, & \text{if } m|k, \\ -1, & \text{if } m \not| k, \end{cases} \tag{4.3}$$

for $k \in \mathbb{Z}$.

Using definition (1.1) (or (2.2)), the polynomial

$$A_\ell(x^m) - \left(\frac{1+x+x^2+\cdots+x^{m-1}}{m}\right)^{\ell+1} A_\ell(x)$$

can be expressed as

$$(1-x^m)^{\ell+1}\sum_{k=1}^{\infty}k^\ell x^{mk}-\left(\frac{1+x+x^2+\cdots+x^{m-1}}{m}\right)^{\ell+1}(1-x)^{\ell+1}\sum_{k=1}^{\infty}k^\ell x^k$$
$$=\left(\frac{1-x^m}{m}\right)^{\ell+1}\cdot\left\{m\cdot\sum_{k=1}^{\infty}(mk)^\ell x^{mk}-\sum_{k=1}^{\infty}k^\ell x^k\right\}.$$

It is enough to show that

$$P(x):=(1+x+\cdots+x^{m-1})^{\ell+1}\left\{m\cdot\sum_{k=1}^{\infty}(mk)^\ell x^{mk}-\sum_{k=1}^{\infty}k^\ell x^k\right\}\tag{4.4}$$

becomes a polynomial. Applying (4.3), we have

$$\begin{aligned}P(x)&=(1+x+\cdots+x^{m-1})^{\ell+1}\sum_{k=1}^{\infty}\sum_{j=1}^{m-1}\zeta_m^{jk}k^\ell x^k\\&=\prod_{i=1}^{m-1}(1-\zeta_m^i x)^{\ell+1}\cdot\sum_{k=1}^{\infty}\sum_{j=1}^{m-1}k^\ell(\zeta_m^j x)^k\\&=\sum_{j=1}^{m-1}\left(\prod_{\substack{1\le i\le m-1\\ i\neq j}}(1-\zeta_m^i x)^{\ell+1}\right)\cdot A_\ell(\zeta_m^j x),\end{aligned}$$

which is a polynomial in x. This completes the proof of Theorem 1.1.

Remark. The congruence (1.2) is not optimal when ℓ is even. Indeed, if ℓ is even, the congruence (1.2) holds modulo $(1-x)^{\ell+2}$, which follows from the symmetry $A(\ell,k)=A(\ell,\ell+1-k)$ and $A_\ell(-1)=0$.

5. A characterization of the Eulerian polynomial

In this section, we prove the following.

Theorem 5.1. *Let $f(x)=x^\ell+a_1x^{\ell-1}+\cdots+a_\ell\in\mathbb{C}[x]$ be a monic complex polynomial of degree $\ell>0$. Then, the following are equivalent.*

(a) $f(x)=A_\ell(x)$.

(b) *For any $m\ge 2$, $f(x)$ satisfies the congruence (1.2). Namely,*

$$f(x^m)\equiv\left(\frac{1+x+\cdots+x^{m-1}}{m}\right)^{\ell+1}f(x)\mod (1-x)^{\ell+1}\tag{5.1}$$

is satisfied.

(c) *The congruence for $m=2$ holds, namely,*

$$f(x^2)\equiv\left(\frac{1+x}{2}\right)^{\ell+1}f(x)\mod (1-x)^{\ell+1}$$

is satisfied.

(d) *There exists an integer $m\ge 2$ such that the congruence (5.1) holds.*

Proof. $(a) \Longrightarrow (b)$ is nothing but Theorem 1.1. The implications $(b) \Longrightarrow (c) \Longrightarrow (d)$ are obvious.

Let us assume (d). We shall prove (a). Choose an integer $m \geq 2$ such that (5.1) is satisfied. There exists a polynomial $g(x) \in \mathbb{C}[x]$ that satisfies

$$f(x^m) - \left(\frac{1 + x + \cdots + x^{m-1}}{m}\right)^{\ell+1} \cdot f(x) = (1-x)^{\ell+1} \cdot g(x). \tag{5.2}$$

Note that $\deg g = m\ell + m - \ell - 2 < (m-1)(\ell+1)$. Dividing this equation by $(1-x^m)^{\ell+1}$, we have

$$\frac{f(x^m)}{(1-x^m)^{\ell+1}} - \frac{1}{m^{\ell+1}} \cdot \frac{f(x)}{(1-x)^{\ell+1}} = \frac{g(x)}{(1+x+\cdots+x^{m-1})^{\ell+1}}. \tag{5.3}$$

We expand $\frac{g(x)}{(1+\cdots+x^{m-1})^{\ell+1}}$ into a partial fraction,

$$\frac{g(x)}{(1+\cdots+x^{m-1})^{\ell+1}} = \sum_{j=1}^{m-1} \frac{R_j(x)}{(1-\zeta_m^j x)^{\ell+1}}, \tag{5.4}$$

where $R_j(x) \in \mathbb{C}[x]$ with $\deg R_j(x) \leq \ell$. As is well known in the theory of formal power series [11], there exist polynomials $\alpha(t), \beta_j(t) \in \mathbb{C}[t]$ with $\deg \alpha(t), \deg \beta_j(t) \leq \ell$ such that

$$\frac{f(x)}{(1-x)^{\ell+1}} = \sum_{k=0}^{\infty} \alpha(k) x^k, \text{ and } \frac{R_j(\zeta_m^{-j} x)}{(1-x)^{\ell+1}} = \sum_{k=0}^{\infty} \beta_j(k) x^k. \tag{5.5}$$

Then, the right hand side of (5.3) is

$$\begin{aligned} \text{RHS of (5.3)} &= \sum_{j=1}^{m-1} \sum_{k \geq 0} \beta_j(k) \zeta_m^{jk} x^k \\ &= \sum_{j=1}^{m-1} \sum_{r=0}^{m-1} \sum_{q=0}^{\infty} \beta_j(qm+r) \zeta_m^{j(qm+r)} x^{qm+r} \\ &= \sum_{q=0}^{\infty} \sum_{r=0}^{m-1} \left(\sum_{j=1}^{m-1} \zeta_m^{jr} \beta_j(qm+r) \right) x^{qm+r}. \end{aligned} \tag{5.6}$$

On the other hand, the left hand side of (5.3) is

(5.7)

$$\begin{aligned} \text{LHS of (5.3)} &= \sum_{k \geq 0} \alpha(k) x^{km} - \frac{1}{m^{\ell+1}} \sum_{k \geq 0} \alpha(k) x^k \\ &= \sum_{q=0}^{\infty} \left(\alpha(q) - \frac{1}{m^{\ell+1}} \alpha(mq) \right) x^{mq} - \sum_{q=0}^{\infty} \sum_{r=1}^{m-1} \frac{\alpha(mq+r)}{m^{\ell+1}} x^{mq+r}. \end{aligned}$$

Comparison of (5.6) and (5.7) gives

$$
\begin{aligned}
\sum_{j=1}^{m-1} \beta_j(qm) &= \alpha(q) - \frac{1}{m^{\ell+1}}\alpha(mq) \\
\sum_{j=1}^{m-1} \beta_j(qm+1)\zeta_m^j &= -\frac{1}{m^{\ell+1}}\alpha(mq+1) \\
&\vdots \\
\sum_{j=1}^{m-1} \beta_j(qm+m-1)\zeta_m^{j(m-1)} &= -\frac{1}{m^{\ell+1}}\alpha(mq+m-1),
\end{aligned}
\tag{5.8}
$$

for any $q \geq 0$. Since both sides of (5.8) are polynomials in q, we have the following polynomial identities.

$$
\begin{aligned}
\sum_{j=1}^{m-1} \beta_j(t) &= \alpha\left(\frac{t}{m}\right) - \frac{1}{m^{\ell+1}}\alpha(t) \\
\sum_{j=1}^{m-1} \beta_j(t)\zeta_m^j &= -\frac{1}{m^{\ell+1}}\alpha(t) \\
&\vdots \\
\sum_{j=1}^{m-1} \beta_j(t)\zeta_m^{j(m-1)} &= -\frac{1}{m^{\ell+1}}\alpha(t).
\end{aligned}
\tag{5.9}
$$

By summing up all identities in (5.9), we obtain a functional equation

$$
\alpha\left(\frac{t}{m}\right) = \frac{1}{m^{\ell}}\alpha(t).
$$

This relation is satisfied only by the polynomial of the form $\alpha(t) = c_0 \cdot t^{\ell}$, where $c_0 \in \mathbb{C}$. Again comparing (1.1) and (5.5), $f(x) = c_0 \cdot A_\ell(x)$. Since $f(x)$ is a monic, we have $f(x) = A_\ell(x)$. □

References

1. Christos A. Athanasiadis, *Extended Linial hyperplane arrangements for root systems and a conjecture of Postnikov and Stanley*, J. Algebraic Combin. **10** (1999), no. 3, 207–225. MR 1723184
2. Matthias Beck and Sinai Robins, *Computing the continuous discretely*, Undergraduate Texts in Mathematics, Springer, New York, 2007, Integer-point enumeration in polyhedra. MR 2271992
3. L. Carlitz, *q-Bernoulli and Eulerian numbers*, Trans. Amer. Math. Soc. **76** (1954), 332–350. MR 0060538
4. Louis Comtet, *Advanced combinatorics*, enlarged ed., D. Reidel Publishing Co., Dordrecht, 1974, The art of finite and infinite expansions. MR 0460128

5. Dominique Foata, *Eulerian polynomials: from Euler's time to the present*, The legacy of Alladi Ramakrishnan in the mathematical sciences, Springer, New York, 2010, pp. 253–273. MR 2744266
6. Thomas Lam and Alexander Postnikov, *Alcoved polytopes. II*, arXiv:1202.4015 (2012).
7. Peter Orlik and Hiroaki Terao, *Arrangements of hyperplanes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 300, Springer-Verlag, Berlin, 1992. MR 1217488
8. T. Kyle Petersen, *Eulerian numbers*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser/Springer, New York, 2015, With a foreword by Richard Stanley. MR 3408615
9. Alexander Postnikov and Richard P. Stanley, *Deformations of Coxeter hyperplane arrangements*, J. Combin. Theory Ser. A **91** (2000), no. 1-2, 544–597, In memory of Gian-Carlo Rota. MR 1780038
10. John Riordan, *An introduction to combinatorial analysis*, Wiley Publications in Mathematical Statistics, John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1958. MR 0096594
11. Richard P. Stanley, *Enumerative combinatorics. Volume 1*, second ed., Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012. MR 2868112
12. Richard P. Stanley and Fabrizio Zanello, *Some asymptotic results on q-binomial coefficients*, Ann. Comb. **20** (2016), no. 3, 623–634. MR 3537923
13. J. Worpitzky, *Studien über die Bernoullischen und Eulerschen Zahlen*, J. Reine Angew. Math. **94** (1883), 203–232. MR 1579945
14. Masahiko Yoshinaga, *Characteristic polynomials of Linial arrangements for exceptional root systems*, J. Combin. Theory Ser. A **157** (2018), 267–286. MR 3780415
15. ———, *Worpitzky partitions for root systems and characteristic quasi-polynomials*, Tohoku Math. J. (2) **70** (2018), no. 1, 39–63. MR 3772805

Department of Mathematics, Hokkaido University, North 10, West 8, Kita-ku, Sapporo 060-0810, JAPAN
E-mail address: k.iijima1203@gmail.com

Department of Mathematics, Hokkaido University, North 10, West 8, Kita-ku, Sapporo 060-0810, JAPAN
E-mail address: kyo.sasaki2308@gmail.com

Department of Mathematics, Hokkaido University, North 10, West 8, Kita-ku, Sapporo 060-0810, JAPAN
E-mail address: sin3xcos4x@gmail.com

Department of Mathematics, Hokkaido University, North 10, West 8, Kita-ku, Sapporo 060-0810, JAPAN
E-mail address: yoshinaga@math.sci.hokudai.ac.jp